An introduction to G_2 geometry via the Bryant-Salamon metrics

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Abstract

The first explicit examples of metrics with exceptional holonomy were found on the total spaces of vector bundles. Bryant and Salamon constructed the first complete G_2 and $Spin(7)$ metrics in this way [BS89], thus removing from doubt the existence of the final cases on Berger's list. They showed that there exists a complete G_2 metric on the bundle of anti-self-dual 2-forms over a compact 4-dimensional manifold if the base is self-dual, Einstein and has positive scalar curvature.

Earlier this year, Herfray-Krasnov-Scarinci-Shtanov generalised this construction to an $SO(3)$-bundle with compatible connection over any 4-dimensional base, so long as the connection satisfies a certain PDE. This PDE is also the field equation for a particular 4-dimensional theory of gravity, and their work was largely motivated by its relevance to theoretical physics. We give an overview of the original Bryant-Salamon construction, before discussing the mathematical detail behind the recent, more general construction of [HKSS16].

Calibrated submanifolds arise naturally in manifolds with special holonomy. Harvey and Lawson found a bundle construction for special Lagrangians in a Calabi-Yau manifold, and this was generalised by Karigiannis and Min-Oo to associative and co-associative submanifolds of the Bryant-Salamon G_2 bundles [KMO05]. We will look in detail at the conditions required for subbundles over immersed surfaces in the base to be calibrated, and will consider the generalisation of this to the recent $SO(3)$-bundle G_2 construction.

Any 'questions' are simply my own, and as such the answers may well already be known to other people. I have been a bit inconsistent with summation notation, but as a rule repeated indices are to be summed over as per the summation convention.

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1 Background

1.1 Holonomy

We briefly review the concept of holonomy, in order to motivate our later discussions.

Definition 1.1. Let M be an n-dimensional manifold, g a Riemannian metric on M, and ∇ the Levi-Civita connection of g. The *holonomy group* $\mathrm{Hol}(g)$ of g is the group of isometries of T_xM generated by parallel transport (as defined by ∇) around closed loops based at $x \in M$. Since $\nabla g = 0$, $\mathrm{Hol}(g)$ is a subgroup of $O(n)$, and is independent of the base point x, up to conjugation in $O(n)$.

A natural question to ask is: which subgroups of $O(n)$ can be the holonomy group of a Riemannian metric g on a manifold M? This question was answered by Berger in 1955, after imposing several restrictions. Firstly that M is simply-connected, to avoid trouble with the global topology. Secondly that g is irreducible, *i.e.* $\mathrm{Hol}(g)$ is not the product of lower-dimensional holonomy groups. And thirdly that g is nonsymmetric (Riemannian symmetric spaces were classified completely by Cartan in 1925, [Car26]). Berger produced the following celebrated list of possible holonomy groups.

Theorem 1.2 (Berger, [Ber55]). *Suppose g is an irreducible, nonsymmetric Riemannian metric on a simply-connected n-dimensional manifold M. Then exactly one of the following is true:*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,

(iii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,

(iv) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,

(v) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Berger showed that these seven possibilities are the only ones (after discarding Spin(9) from his original list since it leads to necessarily symmetric manifolds). He did not show that all of them actually exist as the holonomy groups of a Riemannian metric, but it is now known that they do. Many of the first explicit examples of (non-flat) metrics with special holonomy were constructed on the total spaces of vector bundles. Despite being non-compact, these bundle examples are very useful since they provide local models for a general metric of special holonomy. They are all cohomogeneity one metrics, which is why the special holonomy condition can be reduced to a (solvable) ODE.

Calabi-Yau (holonomy $SU(n)$) metrics on T^*S^n were found first by Eguchi-Hanson for S^2 [EH79] and then Stenzel for the general case [Ste93]. Hyperkähler (holonomy $Sp(m)$) metrics on $T^*\mathbb{CP}^n$ were discovered by Calabi [Cal79]. G_2 and $Spin(7)$ are known as the *exceptional* holonomy groups, and were the last to be shown to exist. In 1989, Bryant and Salamon found complete metrics of full holonomy G_2 and Spin(7) on the bundles of anti-self-dual 2-forms and negative chirality spinors over certain 4-manifolds [BS89]. Their construction depends quite heavily on the geometry of the 4-dimensional base X; the metrics are only complete when X is the self-dual Einstein manifold S^4 or $\mathbb{CP}^2$.

1.2 Self-duality

Definition 1.3. On an oriented Riemannian n-manifold (X, g), the *Hodge star operator* is a homomorphism

$$* : \Lambda^r T^*X \to \Lambda^{n-r} T^*X$$

between forms of complementary index, defined by comparing the natural metric on the forms with the wedge product

$$\alpha \wedge *\beta = (\alpha, \beta) v_g$$

where $(\cdot, \cdot)$ is the inner product defined by the induced metric on $\Lambda^r T^*X$, and v_g is the volume form on X. If $\{e^1, \dots, e^n\}$ is an oriented orthonormal (oON) basis of T^*X, it therefore follows that

$$*(e^1 \wedge \cdots \wedge e^r) = e^{r+1} \wedge \cdots \wedge e^n.$$

Note that we have $*^2 = (-1)^{r(n-r)}$. In even dimensions, $n = 2m$, the Hodge star induces an automorphism of $\Lambda^m X$. This is an involution if m is even, and a complex structure if m is odd.

Definition 1.4. On a 4-manifold the $*$ operator defines an automorphism on 2-forms, such that $*^2 = \mathrm{Id}_{\Lambda^2}$. The *self-dual* (SD) and *anti-self-dual* (ASD) forms, denoted Λ^2_+ and Λ^2_- respectively, are the rank-3 vector bundles defined to be the $+1$ and -1 eigenspaces of $*$. We then have the splitting

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

with $\alpha \wedge \alpha = \pm|\alpha|^2 v_g$ for $\alpha \in \Lambda^2_\pm$.

For any oriented orthonormal basis $\{e^1, \dots, e^4\}$ of T^*X, we have that $\{\psi^i\}, \{\omega^i\}$ given by

$$\begin{aligned}
\psi^1 &= e^1 \wedge e^2 + e^3 \wedge e^4 & \qquad \omega^1 &= e^1 \wedge e^2 - e^3 \wedge e^4 \\
\psi^2 &= e^1 \wedge e^3 + e^4 \wedge e^2 & \qquad \omega^2 &= e^1 \wedge e^3 - e^4 \wedge e^2 \\
\psi^3 &= e^1 \wedge e^4 + e^2 \wedge e^3 & \qquad \omega^3 &= e^1 \wedge e^4 - e^2 \wedge e^3
\end{aligned} \tag{1}$$

are oON (up to a constant) bases of Λ^2_+ and Λ^2_- respectively.

Remark 1.5. An oriented 4-manifold has a natural $SO(4)$ structure. The map from Λ^2 to $\mathfrak{so}(4)$ maps Λ^2_+ and Λ^2_- onto the two 3-dimensional commuting ideals in $\mathfrak{so}(4)$, isomorphic to $\mathfrak{so}(3)$. Thus the splitting of Λ^2 is equivalent to the Lie algebra splitting $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, which corresponds to the fact that the homomorphism $SO(4) \to SO(3) \times SO(3)$ is a double cover. This can be seen explicitly as $\{e^i\} \mapsto (\{\phi^i\}, \{\psi^i\})$.

The curvature operator $R : \Lambda^2 \to \Lambda^2$ may be written in block diagonal form relative to the direct sum decomposition:

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

R is self-adjoint (by the Bianchi identity), so $A = A^\dagger$, $C = B^\dagger$, and $D = D^\dagger$. R then decomposes into its irreducible components as follows

$$\operatorname{tr} A = \operatorname{tr} D = \frac{s}{4} \qquad B = r - \frac{1}{4} s g \qquad \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} - \frac{s}{12} = W$$

where s is scalar curvature, r is the Ricci tensor, B is the tracefree Ricci tensor, and W is the Weyl curvature tensor. The two Weyl components, $W^+ := A - \frac{s}{12}$ and $W^- := D - \frac{s}{12}$ are called the *self-dual* and *anti-self-dual* parts respectively.

Definition 1.6. Recall that a metric g on a manifold X is said to be *Einstein* if its Ricci curvature tensor is proportional to the metric, *i.e.* $r = \lambda g$ for some *Einstein constant* $\lambda \in \mathbb{R}$.

Definition 1.7. When the Weyl tensor vanishes, we say that a manifold is *conformally flat*, and we have that at every point the metric is conformal to a flat metric (that is, there exists a nowhere-vanishing function f such that $g^{ij} = f\delta^{ij}$). Manifolds for which one of W^+ or W^- vanishes are called *half-conformally flat*, or more specifically *self-dual* ($W^- = 0$) and *anti-self-dual* ($W^+ = 0$).

Manifolds with constant sectional curvature are conformally flat, and in fact any simply-connected conformally flat closed Riemannian manifold is conformally equivalent to the standard sphere (*i.e.* there exists a diffeomorphism between them which pulls back one metric to within a conformal factor of the other).

Example 1.8. Some 4-manifold examples: S^4 is conformally flat, $\mathbb{CP}^2$ and K3 surfaces are half-conformally flat. S^4 and $\mathbb{CP}^2$ (with the round and Fubini-Study metrics respectively) are the only such ones which are also Einstein and have positive scalar curvature [Bes08].

The splitting of two-forms also extends to the curvature tensor F of a connection ∇ on a bundle E over X. We write

$$F = F^+ \oplus F^- \in \Gamma\left(\Lambda^2_+ X \otimes \mathfrak{g}_E\right) \oplus \Gamma\left(\Lambda^2_- X \otimes \mathfrak{g}_E\right)$$

where $\mathfrak{g}_E$ is the adjoint bundle, and call a connection *self-dual* or *anti-self-dual* when F^- and F^+ vanish respectively. We can now easily see the following result.

Theorem 1.9 (Singer, [AHS78])**.** *A 4-manifold is Einstein iff the Levi-Civita connection on Λ^2_- is anti-self-dual.*

Proof. The curvature of ∇ on Λ^2_- is given by the bottom row of the block decomposition of the curvature tensor. Thus the self-dual part is $B^\dagger$, which vanishes iff X is Einstein. □

Definition 1.10. A conformal structure $[g]$ on X is an equivalence class of Riemannian metrics on X, under the equivalence relation $g \sim fg$ for all smooth, positive, real functions f on X. Equivalently, a conformal structure is a $CO(n)$-structure on X, where $CO(n) = \mathbb{R}^+ \times SO(n)$ is the conformal group, *i.e.* the stabiliser of a generic class $[g]$.

Claim 1.11. *Over a 4-manifold (X, g) the $*$ operator on 2-forms, and hence also the self-dual and anti-self-dual subspaces, depend only on the conformal class of the Riemannian metric.*

If we have a conformal structure $[g]$ on T_xX, we use the Hodge star to define

$$\Lambda^2 T^*_x X = \Lambda^2_+ \oplus \Lambda^2_-$$

From the other perspective, a conformal structure is defined by a splitting of Λ^2 into these subspaces. Since one is the orthogonal complement of the other, knowing one (Λ^2_+ say) suffices to determine a conformal structure. Picking an oON basis $\{\psi^i\}$ for Λ^2_+ determines a metric on X with respect to which there exists an oON basis $\{e^1, e^2, e^3, e^4\}$ of T^*X satisfying the relations (1) (up to multiplication by a constant), and the conformal class of this metric is independent of the initial choice.

Note that this means all of the above SD and ASD notions are conformally invariant; they depend only on the conformal class of the Riemannian metric on the base space. This correspondence between splittings of Λ^2 and conformal classes of metrics on the base will underpin some of our later constructions, so we prove it in the next two propositions.

Proposition 1.12. *A volume form v on X defines a metric of type $(3, 3)$ on $\Lambda^2 T^*_x X$.*

Proof. Choose a volume form $v \in \Lambda^4 T^*_x X$, then $v = e^1 \wedge e^2 \wedge e^3 \wedge e^4$ defines some basis $\{e^i\}$ of $T^*_x X$. Now define a bilinear form $\langle \cdot, \cdot \rangle$ on $\Lambda^2 T^*_x X \cong \mathbb{R}^6$ by

$$\alpha \wedge \beta = \langle \alpha, \beta \rangle v$$

for $\alpha, \beta \in \Lambda^2 T_x^* X$. Then, with respect to the basis $\{\psi^i\}, \{\omega^i\}$ of Λ^2 given earlier, we have

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Hence this bilinear form defines a metric of type $(3,3)$ on $\Lambda^2 T_x^* X$. □

Proposition 1.13 ([DK90]). *Any positive-definite 3-dimensional subspace of $\Lambda^2 T_x^* X$ uniquely determines a conformal structure on $T_x X$.*

Proof. $SL(4,\mathbb{R})$ acts on $(\mathbb{R}^4)^*$, and the induced action on $\Lambda^2 T_x^* X$ is given by

$$g \cdot (\alpha \wedge \beta) = (g \cdot \alpha) \wedge (g \cdot \beta).$$

Since $SL(4,\mathbb{R})$ preserves the volume form, it must preserve the $(3,3)$-metric on 2-forms (*cf.* the previous proposition). This gives a group homomorphism

$$f : SL(4,\mathbb{R}) \to O(3,3)$$

whose image is actually in $SO(3,3)$ since $SL(4,\mathbb{R})$ is connected (in fact, this map is a double-covering). We can see from this that $SO(4) \to SO(3) \times SO(3)$ is also a double cover, by taking maximal compact subgroups.

Now the subgroup that preserves a positive (or negative) definite subspace L is isomorphic to $SO(3) \times SO(3)$ so

$$G = \frac{SO(3,3)}{SO(3) \times SO(3)}$$

parametrises the set of all positive definite 3-dimensional subspaces of $\Lambda^2 T_x^* X$. It follows that $f^{-1}(SO(3) \times SO(3))$ is a subgroup of $SL(4,\mathbb{R})$ isomorphic to $SO(4)$. Now

$$V = \frac{SL(4,\mathbb{R})}{SO(4)} \cong \frac{GL^+(4,\mathbb{R})}{CO(4)}$$

where $CO(4) = \mathbb{R}^+ \times SO(4)$ is the group of conformal linear transformations of $\mathbb{R}^4$. So V parametrises *reductions* to $CO(4)$ in the presence of a volume form *i.e.* conformal structures.

Given a reduction with metric g, we map this to the subspace Λ^2_+, defined by the Hodge star associated to that g. This mapping $V \to G$ now shows that for each positive definite 3-dimensional subspace there is a unique conformal structure. □

Remark 1.14. Suppose L is a positive-definite 3-dimensional subspace, then set $\Lambda^2_+ = L$ and define

$$\Lambda^2_- = \{\beta \in \Lambda^2 T_x^* X \ : \ \alpha \wedge \beta = 0 \ \ \forall \alpha \in L\}.$$

We now have a splitting $\Lambda^2 T_x^* X = \Lambda^2_+ \oplus \Lambda^2_-$ with induced metric on $\Lambda^2_\pm$ and so on $T_x X$.

Remark 1.15. The conformal structure $[g]$ arising from a positive definite subspace L is exactly the one which makes $L = \Lambda^2_+$ (for the splitting defined by $*_g$).

1.3 G_2-structures

Let $(x_1, \dots x_7)$ be coordinates on $\mathbb{R}^7$, and define the 3-form φ_0 to be

$$\varphi_0 = dx_{123} + dx_{145} - dx_{167} + dx_{246} - dx_{257} + dx_{347} - dx_{356}$$

where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The subgroup of $GL(7,\mathbb{R})$ that preserves φ_0 is the exceptional Lie group G_2 [Bry87]. It is a compact, connected, simply-connected, semisimple, 14-dimensional subgroup of $SO(7)$, which also fixes the Euclidean metric $g_0 = dx_1^2 + \ldots + dx_7^2$, the orientation on $\mathbb{R}^7$, and the 4-form

$$*\varphi_0 = dx_{4567} - dx_{2345} + dx_{2367} - dx_{3146} + dx_{3175} - dx_{1247} + dx_{1256}$$

which is the Hodge dual of φ_0 with respect to g_0.

Remark 1.16. In fact, the ring of G_2-invariant forms on $\mathbb{R}^7$ is spanned by $\{1, \varphi_0, *\varphi_0, *1\}$.

Definition 1.17. A 3-form $\varphi \in \Lambda^3\mathbb{R}^7$ is called *stable* if it lies in an open orbit under the action of $GL(7, \mathbb{R})$.

For real 3-forms, there are two distinct open orbits (each related to a real form of $G_2^{\mathbb{C}}$), which can be distinguished by the sign of an invariant. We will always consider the positive orbit, corresponding to the compact real form G_2.

Suppose M is a 7-dimensional Riemannian manifold. For each $p \in M$ define $\mathcal{P}_p^3$ to be the 3-forms $\varphi \in \Lambda^3 T_p^* M$ for which there exists an oriented isomorphism $f : T_pM \to \mathbb{R}^7$ with $\varphi = f^*\varphi_0$ (*i.e.* the $GL^+(7, \mathbb{R})$-orbit of φ_0). Then $\mathcal{P}_p^3 M$ is isomorphic to $GL^+(7, \mathbb{R})/G_2$, so has dimension $49 - 14 = 35$. But $\Lambda^3 T_p^* M$ also has dimension 35, so $\mathcal{P}_p^3 M$ is indeed an open subset of $\Lambda^3 T_p^* M$.

Let $\mathcal{P}^3 M$ be a fibre bundle over M, with fibre $\mathcal{P}_p^3 M$. Then $\mathcal{P}^3 M$ is an open subbundle of $\Lambda^3 T^* M$ with fibre $GL^+(7, \mathbb{R})/G_2$ (not a vector subbundle though).

Definition 1.18. We call a 3-form φ *positive* if $\varphi|_p \in \mathcal{P}_p^3 M$ for all $p \in M$.

Remark 1.19. Note that for all positive 3-forms, φ, there exist 1-forms $\theta_1, \ldots, \theta_7$ such that φ can be written in the canonical form given for φ_0 above.

Definition 1.20. The *frame bundle* F of M is the bundle over M whose fibre at $p \in M$ is the set of all isomorphisms $T_pM \to \mathbb{R}^7$. For φ a positive 3-form, let Q be the subset of F consisting of the isomorphisms which send $\varphi|_p \mapsto \varphi_0$. Now Q is a principal subbundle of F, with fibre G_2, and we call it a *G_2-structure*.

Conversely, a G_2-structure Q on M defines a 3-form φ, a 4-form $*\varphi$, and a metric g (such that $\varphi_0, *\varphi_0$ and g_0 are G_2-invariant). The 3-form φ will be positive iff Q is positively oriented (*i.e.* induces the given orientation on M).

Furthermore, any positive 3-form φ defines a unique positive 4-form $*\varphi$ and metric g such that every tangent space of M admits an isomorphism with $\mathbb{R}^7$ sending $\varphi, *\varphi$ and g to $\varphi_0, *\varphi_0$ and g_0 respectively. For M an oriented 7-manifold and φ a positive 3-form on M, the associated metric g is given by

$$g_\varphi(\xi, \eta) v_g = \iota_\xi \varphi \wedge \iota_\eta \varphi \wedge \varphi$$

where ξ, η are tangent vector fields, $\iota_\xi \varphi := \varphi(\xi, \cdot)$, and v_g is a volume form whose sign is determined by the requirement that the metric has a Euclidean signature. When the 3-form is given in canonical form as before, the metric is

$$g_\varphi = \sum_{i=1}^{7} \theta^i \otimes \theta^i$$

and the volume form simply $\theta^1 \wedge \cdots \wedge \theta^7$. We often call the pair (φ, g_φ) a *G_2-structure* on M (in a slight abuse of the definition given previously).

Remark 1.21. The functional defined by integrating the volume form over the manifold

$$S[\varphi] = \int_M v_{g_\varphi}$$

is non-zero iff φ is stable, and its sign characterises which $GL(7, \mathbb{R})$ orbit φ lies in.

Definition 1.22. Let M be a 7-manifold, and (φ, g) a G_2-structure on M. Then $\nabla\varphi$ is the *torsion* of (φ, g), for ∇ the Levi-Civita connection of g. A *G_2-manifold* is defined to be a triple (M, φ, g) such that (φ, g) is a *torsion-free*, *i.e.* $\nabla\varphi = 0$.

The following theorem characterises G_2-manifolds, and will be crucial to the construction of G_2 metrics later on.

Theorem 1.23 (Gray, [FG82])**.** *Let M be a 7-manifold, and (φ, g) a G_2-structure on it. Then the following are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq G_2$*, and φ is the induced 3-form by the inclusion,*

(ii) $\nabla\varphi = 0$ *on M, for ∇ the Levi-Civita connection of g,*

(iii) $d\varphi = d^*\varphi = 0$ *on M,*

(iv) $d\varphi = d(*\varphi) = 0$ *on M.*

Don't be fooled into thinking that any of these conditions are linear in φ; both ∇ and $*$ depend on g, which is determined by φ. The condition on φ is actually a nonlinear PDE.

Remark 1.24. Combining this theorem with the first variation formula for the functional defined above

$$\delta S[\varphi] \sim \int_M *\varphi \wedge \delta\varphi$$

we see that G_2-manifolds are critical points of $S[\varphi]$, as long as we vary φ in a fixed cohomology class $\delta\varphi = dB$, $B \in \Lambda^2 M$. We will mention this relationship again later, but for a more rigorous account, see [Hit00].

An important well-known property of G_2-manifolds is the following:

Theorem 1.25 (Bonan, [Bon66])**.** *Let (M, g) be a Riemannian 7-manifold. If* $\mathrm{Hol}(g) \subseteq G_2$*, then g is Ricci-flat.*

In fact we could go further, and say that since $G_2 \subset SO(7)$ is simply-connected, any manifold with a G_2-structure is spin, but we shan't be concerned with spin structures here Manifolds for which one of W^+ or W^- vanishes are called *half-conformally flat*, or more specifically *self-dual* ($W^- = 0$) and *anti-self-dual* ($W^+ = 0$).

We have seen equivalent conditions for a metric g on M to have holonomy *contained* in G_2, but what if we ask that the holonomy is *equal* to G_2? A simple criterion for full holonomy G_2 is that $\{1, \varphi, *\varphi, *1\}$ span the full set of ∇-parallel forms on M (*cf.* Remark 1.16). In fact, Bryant proved a stronger result.

Theorem 1.26 (Bryant, [Bry87])**.** *Let M be a simply-connected, connected manifold, and g be a Riemannian metric on M whose holonomy is a subgroup of G_2. Then the holonomy of g is equal to G_2 iff there are no non-zero ∇-parallel 1-forms on M.*

2 The Bryant-Salamon metrics

2.1 The G_2-structure on $\Lambda^2_- T^*X$

Suppose (X, g) is an oriented Riemannian 4-manifold and let $M = \Lambda^2_- T^*X$ be the bundle of anti-self-dual 2-forms on X. M is isomorphic to $\mathbb{R}^7$, with a natural G_2-structure. Suppose $\{e^1, e^2, e^3, e^4\}$ is an oriented coframe of orthonormal covector fields on T^*X. Then the following is a basis of sections for $M = \Lambda^2_- T^*X$

$$\omega^1 = e^1 \wedge e^2 - e^3 \wedge e^4 \qquad \omega^2 = e^1 \wedge e^3 - e^4 \wedge e^2 \qquad \omega^3 = e^1 \wedge e^4 - e^2 \wedge e^3$$

The Levi-Civita connection on X induces an $SO(3)$-connection ∇ on M, which is the ASD part of the Levi-Civita connection, satisfying $\nabla\omega^i = 0$. This induces a canonical splitting of the tangent space into horizontal and vertical subspaces, $T_\omega M \cong \mathcal{H}_\omega \oplus \mathcal{V}_\omega$ at a point $\omega \in \Lambda^2_-$. The projection map is a submersion which maps the horizontal space isometrically onto the tangent space of the base at that point. The metric g on X has a unique lift to the horizontal space, $g_\mathcal{H}$, and g also induces a natural metric, $g_\mathcal{V}$, on the vertical space (which can be identified with the fibre $\Lambda^2_- T^*_x X$).

An element of the total space $\Lambda^2_- T^*X$ can be written $(x, t_1\omega^1 + t_2\omega^2 + t_3\omega^3)$, for $x \in X$, t_i fibre coordinates. Denote by β_i the horizontal lift $(e_i, 0)$ of tangent vector e_i (dual to e^i) to $\mathcal{H}$, and by α_i the vertical tangent vector $(0, \omega^i)$ in $\mathcal{V}$ on the total space. We see that $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\}$ is an orthonormal tangent frame for the total space.

Then the canonical G_2 form φ on $\Lambda^2_- X$ is a 3-form on the total space, locally given by

$$\begin{aligned}\varphi = {} & \alpha^1 \wedge \alpha^2 \wedge \alpha^3 + \alpha^1 \wedge (\beta^1 \wedge \beta^2 - \beta^3 \wedge \beta^4) \\ & + \alpha^2 \wedge (\beta^1 \wedge \beta^3 - \beta^4 \wedge \beta^2) + \alpha^3 \wedge (\beta^1 \wedge \beta^4 - \beta^2 \wedge \beta^3)\end{aligned}$$

where α^i and β^j are dual to α_i and β_j respectively. In the same basis, the dual 4-form is given by

$$\begin{aligned}*\varphi = {} & \beta^1 \wedge \beta^2 \wedge \beta^3 \wedge \beta^4 - \alpha^2 \wedge \alpha^3 \wedge (\beta^1 \wedge \beta^2 - \beta^3 \wedge \beta^4) \\ & - \alpha^3 \wedge \alpha^1 \wedge (\beta^1 \wedge \beta^3 - \beta^4 \wedge \beta^2) - \alpha^1 \wedge \alpha^2 \wedge (\beta^1 \wedge \beta^4 - \beta^2 \wedge \beta^3).\end{aligned}$$

Remark 2.1. Note that these forms are actually the same as the canonical form in Section 1.3; the ASD bits arise naturally in G_2-structures, in relation to the embedding of $SO(3)$ in $SO(4) \subset G_2$.

2.2 The Bryant-Salamon construction

Robert Bryant showed in [Bry87] that there exists a metric with full holonomy G_2 on an open set of $\mathbb{R}^7$ (and a Spin(7) one on $\mathbb{R}^8$), thus settling the remaining cases of Berger. He did so by reformulating the 'holonomy H' condition as a set of differential equations for the associated H-structure on a manifold, and then applying the machinery of over-determined partial differential equations (in the form of the Cartan-Kählertheorem) to prove the existence of solutions with holonomy G_2 (or Spin(7)).

He included an example of a G_2 metric on $\mathbb{R}^+ \times M$, where M is a certain six-dimensional homogeneous space. A deeper understanding of this example led to Bryant and Salamon explicitly constructing three distinct *complete* metrics with holonomy equal to G_2 [BS89]. We focus on the ones which are encountered on the total spaces of vector bundles over 4-manifolds. In this section we give an overview of their result, highlighting the important ingredients, but leave much of the technical detail for the more general construction in Section 3.

Theorem 2.2 (Bryant-Salamon, [BS89])**.** *There exist positive functions f and g of r (the radial coordinate in the fibres), satisfying a certain set of ODEs, such that the metric*

$$g_M = f^2 g_\mathcal{H} \oplus g^2 g_\mathcal{V}$$

on the total space $M = \Lambda^2_-(T^*X)$ of the bundle of anti-self-dual 2-forms on a self-dual Einstein 4-manifold (X, g) has full holonomy G_2, with associated 3-form φ given by

$$\varphi = g^3 \operatorname{vol}_\mathcal{V} + f^2 g d\theta$$

where θ is the canonical 2-form on $\Lambda^2_-(T^*X)$ and $\operatorname{vol}_\mathcal{V}$ is the volume 3-form of $g_\mathcal{V}$ on the vertical fibres.

Definition 2.3. The *canonical p-form* on $\Lambda^p(T^*X)$ is $\theta_\omega(u_1 \wedge \ldots \wedge u_p) = \omega(\pi_* u_1 \wedge \ldots \wedge \pi_* u_p)$ at the point ω, where π is the projection onto the base X. For $p = 1$ this is the more familiar canonical 1-form $\theta : T^*X \to T^*(T^*X)$ defined by $\theta_\alpha(u) = \alpha(\pi_* u)$, where $\alpha \in T^*X$ and $u \in T_\alpha(T^*X)$.

Bryant and Salamon constructed their full holonomy G_2 metrics using criteria we encountered earlier. Namely, they constructed a positive 3-form φ which is closed and co-closed, and then showed that there are no ∇-parallel 1-forms on M (where ∇ is the Levi-Civita connection of g_φ). The metrics are constructed on the bundle of ASD 2-forms, where they exploit the fact that for a self-dual Einstein base X, the curvature of $\Lambda^2_- X$ is entirely determined by the scalar curvature of the base.

Fact 1. *(X, g) is self-dual and Einstein iff there is a constant κ ($= \frac{1}{12}$ scalar curvature) such that*

$$F^i = \kappa\omega^i$$

where F is the curvature of ∇, and ω^i is the canonical basis of ASD 2-forms.

In the notation of the previous subsection, we can take $\mathrm{vol}_\mathcal{V} = \alpha^1 \wedge \alpha^2 \wedge \alpha^3$, and write $d\theta = \alpha^1 \wedge \omega^1 + \alpha^2 \wedge \omega^2 + \alpha^3 \wedge \omega^3$ (where technically the ω's are now defined in terms of the β's). There will be more detail on all these forms in the next section. We have the following ansatz for the 3-form

$$\varphi = f^3\alpha^1 \wedge \alpha^2 \wedge \alpha^3 + 2fg^2(\alpha^1 \wedge \omega^1 + \alpha^2 \wedge \omega^2 + \alpha^3 \wedge \omega^3)$$

where f and g are positive smooth functions of r, corresponding to scalings in the fibres and on the base respectively. This 3-form is both positive and well-defined on all of $\Lambda^2_- X$. It can then be shown that $d\varphi = d * \varphi = 0$ is equivalent to the following overdetermined system of ODEs:

$$(fg^2)' = \frac{\kappa}{2}f^3 \qquad (f^2g^2)' = 0 \qquad (g^4)' = 2\kappa f^2 g^2$$

This system is compatible and has general solution

$$f(r) = \sqrt{c_1}(2\kappa(c_1 r + c_0))^{-\frac{1}{4}}$$
$$g(r) = (2\kappa(c_1 r + c_0))^{\frac{1}{4}}$$

on the domain $2\kappa(c_1 r + c_0) > 0$, for c_0, c_1 constants ($c_1 > 0$). We can set the constants to be ± 1, since scaling f and g produces metrics in the same conformal class. Using this, and the fact that rescaling the fibres gives essentially equivalent metrics, Bryant and Salamon reduced to 5 different sets of solutions for κ, g and f. Each case gives rise to a 3-form φ and associated metric g_φ on an open subset of $\Lambda^2_- X$. They showed that the functions f and g are globally defined, the metric is complete, and there are no non-zero parallel 1-forms, only in the following case of positive scalar curvature:

$$\kappa > 0 \qquad f(r) = (1+r)^{-\frac{1}{4}} \qquad g(r) = (2\kappa)^{\frac{1}{2}}(1+r)^{\frac{1}{4}} \tag{2}$$

We know that S^4 and $\mathbb{CP}^2$ with the standard metrics are the only complete self-dual Einstein 4-manifolds with positive scalar curvature (*cf.* Example 1.8), and so we reach the following conclusion.

Theorem 2.4 (Bryant-Salamon, [BS89]). *Let (M, g) be either S^4 with the standard round metric or $\mathbb{CP}^2$ with the Fubini-Study metric. Then the metric g_φ on $\Lambda^2_- M$ defined by functions (2) is complete and has holonomy equal to G_2.*

In other cases, *e.g.* hyperbolic space, the functions are not globally defined and the metric is incomplete, only being defined near the zero section of $\Lambda^2_-(T^*X)$. There are many more complete examples to be found, however, if one considers orbifolds (which I am not qualified to do).

3 A more general G_2 bundle construction

The recent paper [HKSS16] generalises the construction of the Bryant-Salamon metrics on the total space of the bundle of anti-self-dual 2-forms over a self-dual Einstein 4-manifold. They again consider vector bundles over a 4-dimensional base, with fibre $\mathbb{R}^3$, and an $SO(3)$ structure corresponding to $\mathrm{SO}(3) \subset G_2$. But this time the 3-form is parametrised by an $SO(3)$-connection, with the ansatz using the curvature 2-forms of the $SO(3)$-connection instead of the ASD ones.

They show that this defines a metric with holonomy contained in G_2 if the connection satisfies a certain second order PDE. This PDE happens to be the same as the field equation of a 4D gravity theory (formulated as a diffeomorphism-invariant theory of $SO(3)$ connections [FKP14]). So every solution of this 4D gravity theory (which is not general relativity, but is in its 'family') can be lifted to a G_2 holonomy metric. This link between a theory of differential forms in 7 dimensions and gravity in 4 dimensions could be interpeted as evidence for M-theory, in which metrics with holonomy G_2 provide the internal geometries for the supersymmetry-preserving compactification down to 4-dimensional space-time.

The authors are motivated by the applications to theoretical physics mentioned above, but we largely ignore this aspect, and instead aim to provide a mathematically detailed account of their construction.

3.1 Connection 1-forms

Let X be an oriented 4-dimensional Riemannian manifold, and $\pi\colon E \to X$ a real vector bundle with fibre $\mathbb{R}^3$ and structure group $SO(3)$ (so we are working with frames that form an oON basis at each point). Since $\mathbb{R}^3 \cong \mathfrak{so}(3)$, the fibres carry a natural metric and orientation, given by the Killing form and Lie bracket respectively. This means that it makes sense to talk about an oriented orthonormal (oON) basis of sections of E, over suitable open subsets of X. Let $\{s_1, s_2, s_3\}$ be such a basis.

Example 3.1. $E = \Lambda^2_- T^*X$ with

$$s_1 = e^{12} - e^{34} \qquad s_2 = e^{13} - e^{42} \qquad s_3 = e^{14} - e^{23}$$

where $\{e^i\}$ is an oON basis of 1-forms on X, and $e^{ij} = e^i \wedge e^j$.

Let ∇ denote a connection on E compatible with the $SO(3)$-structure (meaning that the associated parallel transport map sends $SO(3)$-frames to $SO(3)$-frames). Then

$$\nabla s_i = \sum_j A_i^j \otimes s_j$$

where A_i^j are the *connection 1-forms.*

Remark 3.2. Note that, in a slight abuse of notation, we shall denote both the connection and the covariant derivative it induces by ∇.

Proposition 3.3. $\left(A_i^j\right)$ *is a skew-symmetric matrix,* i.e. $A_i^j + A_j^i = 0$.

Proof. Since the basis is orthonormal, we have $\delta_{ij} = \langle s_i, s_j \rangle$. Take the covariant derivative with respect to an arbitrary vector field ξ to get

$$0 = \nabla_\xi \langle s_i, s_j \rangle = \left\langle \sum_j A_i^j(\xi) s_j, s_j \right\rangle + \left\langle s_i, \sum_k A_j^k(\xi) s_k \right\rangle = A_i^j(\xi) + A_j^i(\xi)$$

□

In keeping with the notation of [HKSS16], we denote A_1^2 by A^3 cyclically (using the Lie bracket). A is a 1-form with values in $\mathfrak{so}(3)$:

$$\left(A_i^j\right) = \begin{pmatrix} 0 & A^3 & -A^2 \\ -A^3 & 0 & A^1 \\ A^2 & -A^1 & 0 \end{pmatrix}$$

Consider a section $s = \sum y^i s_i$ of E over $U \subset X$, where the y^i are (for the moment) smooth functions on U (fibre coordinates). Then

$$\begin{aligned} \nabla s = \nabla\Big(\sum_i y^i s_i\Big) &= \sum_i \left(dy^i \otimes s_i + y^i \nabla s_i\right) \\ &= \sum_i \Big(dy^i \otimes s_i + y^i \sum_j A_i^j \otimes s_j\Big) \\ &= \sum_i \Big(dy^i + \sum_j y^j A_j^i\Big) \otimes s_i. \end{aligned}$$

So we consider the 1-forms on U given by

$$\alpha^i := \nabla y^i = dy^i + y^j A_j^i$$

to give us

$$\nabla s = \sum_i \alpha^i \otimes s_i.$$

Example 3.4.

$$\alpha^1 = dy^1 + \sum_j y^j A_j^1 = dy^1 - y^2 A^3 + y^3 A^2$$

3.2 Curvature 2-forms

Let Γ denote the sheaf of local sections of a vector bundle. A connection on E is a first order differential operator $\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$ that is compatible with the exterior derivative in the manner illustrated above. We can extend the definition of ∇ to arbitrary E-valued forms, thus regarding it as a differential operator on the tensor product of E with the full exterior algebra of forms. In fact, there exists a unique extension

$$\nabla_* : \Gamma(\Lambda^* T^* X \otimes E) \to \Gamma(\Lambda^* T^* X \otimes E)$$

such that ∇_* is a derivation on the sheaf of graded modules $\Gamma(\Lambda^* T^* X \otimes E)$, *i.e.* $\nabla_k(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$. This is called the *extended covariant derivative.*

The *curvature* of ∇ can be defined as the composition

$$F : \Gamma(E) \xrightarrow{\nabla_0} \Gamma(T^* M \otimes E) \xrightarrow{\nabla_1} \Gamma(\Lambda^2 T^* M \otimes E)$$

where ∇_0 just represents ordinary covariant differentiation, and ∇_1 is the extended covariant derivative, satisfying $\nabla_1(\alpha \otimes s) = d\alpha \otimes s - \alpha \wedge \nabla s$. More generally, the curvature measures the failure of the sequence

$$\Gamma(E) \xrightarrow{\nabla_0} \Gamma(T^* X \otimes E) \xrightarrow{\nabla_1} \Gamma(\Lambda^2 T^* X \otimes E) \xrightarrow{\nabla_2} \dots \xrightarrow{\nabla_{n-1}} \Gamma(\Lambda^n T^* X \otimes E)$$

to be a chain complex. So for a local section s of E, we have

$$\begin{aligned}
\nabla_1(\nabla_0\, s) &= \nabla_1\Big(\sum_j \alpha^j \otimes s_j\Big) \\
&= \sum_j \left[d\alpha^j \otimes s_j - \alpha^j \wedge \nabla s_j\right] \\
&= \sum_j \Big[\Big(\sum dy^i \wedge A_i^j + y^i dA_i^j\Big) - \sum \alpha^i \wedge A_i^j\Big] \otimes s_j \\
&= \sum_j \Big[\Big(\sum dy^i \wedge A_i^j + y^i dA_i^j\Big) - \sum \left(dy^i + y^j A_j^i\right) \wedge A_i^j\Big] \otimes s_j \\
&= \sum_j \sum_i y^i \left(dA_i^j - \sum A_i^k \wedge A_k^j\right) \otimes s_j.
\end{aligned}$$

We define the $\mathfrak{so}(3)$-valued *curvature 2-forms* to be

$$F_i^j := dA_i^j - \sum A_i^k \wedge A_k^j.$$

This is actually Cartan's second structure equation (obtained via the method of moving frames). We then have the relation

$$\nabla_1(\nabla s) = \sum_{i,j} y^i F_i^j \otimes s_j.$$

Intuitively we think '$\nabla^2 = F$', but note that F is a tensor not a second order differential operator (since $F(fv) = fF(v)$ for smooth functions f). Defining $F^3 = F_1^2$ cyclically as before, we recover the 2-forms F^i defined in [HKSS16]:

$$F^i = dA^i + \frac{1}{2}\epsilon^{ijk} A^j \wedge A^k.$$

Example 3.5.

$$F^1 = F_2^3 = dA_2^3 - \sum A_2^k \wedge A_k^3 = dA_2^3 - A_2^1 \wedge A_1^3 = dA_1 - A_3 \wedge A_2$$

For fixed y^i, we can interpret s as a point of the total space of E. In this way, the y^i become functions on E. If we identify A_i^j with $\pi^* A_i^j$ then α^i is a genuine 1-form on E. Its restriction to the fibre of E is just dy^i. The annihilator of these three 1-forms in each tangent space T_eE is the *horizontal* subspace H_e defined by ∇. Since if ∇s vanishes in some point $x \in M$ then the submanifold $s(U)$ is tangent to $H_{s(x)}$, and this does not depend on the choice of local basis.

In order to simplify calculations later, we may work over a point $x \in X$ for which the local basis $\{s_i\}$ satisfies $(\nabla s_i)_x = 0$, so that $A^i = 0$ and therefore $d = \nabla$ over x. This is always possible: we only need to choose the oON basis $\{s_i\}$ at x and extend it so that the sections are tangent to the respective horizontals H_{s_i}. With this assumption that $A_j^i = 0$ over x, we get

$$d\alpha^i = \epsilon_{ijk}\, y^j F^k \quad \text{and} \quad dF_j^i = 0 \qquad \text{at } \ x.$$

The former can be seen by $d\alpha^i = d(dy^i + y^j A_j^i) = dy^j A_j^i + y^j dA_j^i = y^j F_j^i = \epsilon_{ijk}\, y^j F^k$. The latter is obvious (since $F_j^i = dA_i^j$ is now exact and so definitely closed), and is actually a reduced form of the second Bianchi identity: $\nabla F = 0$. This always holds, even without our above assumption.

We know that $F : E \to \Lambda^2 T^*X \otimes E$, *i.e.* $F \in \Lambda^2 T^*X \otimes \mathrm{End}(E)$. The extended covariant derivative acts on F

$$F \in \Lambda^2 T^*X \otimes \mathrm{End}(E) \xrightarrow{\nabla_2} \Lambda^3 T^*X \otimes \mathrm{End}(E)$$

so the *second Bianchi identity* says that

$$0 = \nabla F = \sum [dF^i \otimes P^i + F^i \wedge \nabla P^i]$$

for $F = \sum F^i \otimes P^i$ the curvature, and P^i a local basis of the adjoint bundle $\mathfrak{g}_E$. We can see intuitively why this always holds, since for any vector field ξ, we have

$$(\nabla F)\xi = \nabla(F(\xi)) - F(\nabla\xi) = \nabla^3\xi - \nabla^3\xi = 0.$$

Our above assumption that we can work over a point for which $\nabla P^i = 0$ means that we have $\nabla F = 0 \implies dF^i = 0$ at that point.

3.3 Definite connections

First, choose a volume form v on X. Then the equation

$$F^i \wedge F^j = -2X^{ij} v \tag{3}$$

) defines a symmetric matrix, X^{ij} of functions on X.

Remark 3.6. Notice that in the Bryant-Salamon case, we had $F^i = \kappa\omega^i$, and so

$$F^i \wedge F^j = \kappa^2 \omega^i \wedge \omega^j = -2\kappa^2 \delta^{ij} v.$$

Suppose that we can choose v such that X is a *definite* matrix (all eigenvalues have the same sign). We then call ∇ a *definite connection.* By changing the sign of v, we can require that X is positive-definite, and this gives us a preferred orientation on X. Then, by the earlier discussion in Section 1.2, we have that

$$L = \left\langle F^1, F^2, F^3 \right\rangle$$

is a 3-dimensional negative-definite subspace of Λ^2 (negative because of the minus sign in equation (3)). Then by Proposition 1.13 there is a unique conformal structure on M such that $L = \Lambda^2_-$ is the space of ASD 2-forms.

So far, everything we've done has seemed dependent on that first choice of volume form v. But there is a way around that! For $F^i \wedge F^j = -2X^{ij} v$ as above, define

$$v_0 = (\det X)^{\frac{1}{3}} v.$$

Suppose we had initially picked $\tilde{v} = fv$ as our volume form, for f some nowhere-vanishing function. This would define $\tilde{X}^{ij}$ by

$$F^i \wedge F^j = -2\tilde{X}^{ij} \tilde{v} = -2\tilde{X}^{ij} f v$$

so we must have $\tilde{X}^{ij} = \frac{1}{f} X^{ij}$, which gives

$$\tilde{v}_0 = (\det \tilde{X})^{\frac{1}{3}} \tilde{v} = (\tfrac{1}{f^3} \det X)^{\frac{1}{3}} f v = (\det X)^{\frac{1}{3}} v = v_0.$$

Hence v_0 is a volume form which depends only on (F^i). Then we define X_0^{ij} via

$$F^i \wedge F^j = -2X_0^{ij} v_0$$

as a definite 3×3 matrix that depends only on (F^i). Notice that since $v_0 = (\det X_0)^{\frac{1}{3}} v_0$, we must have $\det X_0 = 1$. Since X_0 is definite, diagonalisable, and 3×3, this means that it is always *positive* definite. Hence the (F^i) singlehandedly determine the unique conformal structure $[g]$ such that

$$\langle F^1, F^2, F^3 \rangle = \Lambda^2_-.$$

The volume form v_0 then defines a canonical orientation on X, and so on Λ^2_-. We can also see more abstractly that a splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ induces an orientation on the summands, by the correspondence with the splitting $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$; $\mathfrak{so}(3)$ carries a natural orientation thanks to the Lie bracket.

The fact that $\mathfrak{so}(3)$ carries a natural orientation ($[e_1, e_2] = -2e_3$ say) induces another orientation on $\Lambda^2_- := \langle F^1, F^2, F^3 \rangle$, via the curvature of the connection $F = F^i \otimes e_i$ (by assuming that $e_i \mapsto F^i$ is orientation-preserving). If this orientation agrees with the one induced from the orientation on X (defined above), we say that the *sign* of the definite connection is positive; if they are different then it is negative.

Definition 3.7. Let ∇ be a definite connection in an $SO(3)$-bundle $E \to X$. Suppose (e_1, e_2, e_3) is an oriented local frame of $\mathfrak{so}(E)$, and the curvature of ∇ is $F = F^i \otimes e_i$. Then ∇ is called *positive definite* if (F^1, F^2, F^3) is an oriented basis for Λ^2_- and *negative definite* otherwise. We will encode this in the variable σ, which takes values ± 1 accordingly.

Now by choosing the volume form v_0 itself, we can specify a metric within the conformal class defined by the splitting.

Lemma 3.8. *The following equation distinguishes a metric g (known to physicists as the 'Urbantke metric') on X within the conformal structure defined by L*

$$g(\xi,\eta)v_0 = \tfrac{\sigma}{6}\epsilon_{ijk}(\iota_\xi F^i)\wedge(\iota_\eta F^j)\wedge F^k \tag{4}$$

for tangent vectors ξ,η at $x\in X$.

Proof. As shown above, the volume form is determined by (F^i). The double antisymmetrisation (arising from ϵ_{ijk} and wedging 1-forms) ensures that g is a symmetric tensor. We can check that (4) works if (F^i) is an oON basis of Λ^2_- as in (1), with calculations like

$$g(e_1,e_1)v_0 = \frac{\sigma}{6}\left(e^{2314}+e^{3412}+e^{4213}-e^{2413}-e^{3214}-e^{4312}\right) = \sigma\, e^1\wedge e^2\wedge e^3\wedge e^4.$$

(Note incidentally that when a 2-form is wedged with $F^k\in\Lambda^2_-$ its component in Λ^2_+ is annihilated.) Now for fixed ξ,η, (4) defines a totally antisymmetric mapping $\Lambda^3L\to\mathbb{R}$, i.e. an element of $(\Lambda^3L)^*$. This means that if $A\in GL(3,\mathbb{R})$ then

$$\epsilon_{ijk}\ \iota_\xi(A^i_pF^p)\wedge\iota_\eta(A^j_pF^q)\wedge A^k_rF^r = \epsilon_{ijk}(\det A)\ \iota_\xi F^i\wedge\iota_\eta F^j\wedge F^k$$

and the choice of basis (F^i) is irrelevant up to a conformal factor. □

Remark 3.9. The σ in the metric definition compensates for the fact that you might have to swap the order of wedging F^1,F^2,F^3 in order to make them agree with the natural orientation on Λ^2_-.

Remark 3.10. It can be shown that v_0 coincides with the volume form induced from the metric tensor g it defines (at least up to multiplication by a scalar).

Question 3.11. When does a manifold admit a (compatible) definite connection? This question is difficult, at least for the positive case ($\sigma=1$). Fine-Panov showed that if X^4 has an S^1-action preserving a positive definite connection, then it must be S^4 or $\mathbb{CP}^2$ [FP15].

Question 3.12. What does the space of definite connections look like topologically? It is open in the space of all connections, since any sufficiently small deformation will still have definite curvature, but can we tell anything more?

Remark 3.13. We make some comparisons. In the original Bryant-Salamon construction, we already have a metric on the base X, and we use the 2-forms that are an orthonormal basis of $\Lambda^2_-T^*X$ to construct our G_2 form. In the more general [HKSS16] construction, we have no metric on the base, and instead use the curvature 2-forms of a definite connection on an SO(3)-bundle over X. We have seen that these *define* a conformal class of metrics on the base: the one for which they are a (non-orthonormal) basis of the $\Lambda^2_-T^*X$.

In the Bryant-Salamon case, the fact that the metric on the base is Einstein tells us (Theorem 1.9) that the Levi-Civita connection on Λ^2_- is ASD (F^+ vanishes). In the [HKSS16] construction we *define* Λ^2_- as the span of the F^i_j of our connection, *i.e.* we end up choosing a metric on the base that makes the $SO(3)$-connection ASD.

The Bryant-Salamon construction required a self-dual Einstein base to give the relationship $F^i=\kappa\omega^i$ (*cf.* Fact 1), but the [HKSS16] construction defined the G_2 form using the F^i instead of the ω^i, thus bypassing the need for such a relationship.

3.4 Useful forms and identities

We gather together the following abbreviations for forms defined on the total space E (with summation convention), so that we don't lose track of things:

degree	abbreviation	description	equals
0	r	radial coordinate	$\sum(y^i)^2$
1	$\frac{1}{2}dr$		$y^j\alpha^j$
2	θ	canonical 2−form	y^jF^j
3	$d\theta$		$\alpha^j \wedge F^j$
	γ	fibre volume form	$\alpha^1 \wedge \alpha^2 \wedge \alpha^3$

Lemma 3.14. *We have the following identities, which will be useful in the next section:*

(i) $d\theta = \alpha^j \wedge F^j$ *(as claimed above),*

(ii) $d\gamma = \frac{1}{2}dr \wedge d\theta$.

Proof. (i)
$$\begin{aligned} d\theta = d(y^jF^j) &= (dy^j) \wedge F^j + y^j(dF^j) \\ &= (\alpha^j - y^iA^j_i) \wedge F^j + y^j(dF^j) \\ &= \alpha^j \wedge F^j - y^j(A^i_j \wedge F^i + dF^j) \\ &= \alpha^j \wedge F^j - y^j(\nabla F^j) \\ &= \alpha^j \wedge F^j \end{aligned}$$
by the Bianchi identity.

(ii)
$$\begin{aligned} d\gamma = d(\alpha^1 \wedge \alpha^2 \wedge \alpha^3) &= (y^2F^3 - y^3F^2) \wedge \alpha^2 \wedge \alpha^3 \\ &\quad + \alpha^1 \wedge (-y^3F^1 + y^1F^3) \wedge \alpha^3 \\ &\quad + \alpha^1 \wedge \alpha^2 \wedge (y^1F^2 - y^2F^1) \\ &= \left(\sum y^i\alpha^i\right) \wedge \left(\sum \alpha^j \wedge F^j\right) \\ &= \tfrac{1}{2}\, dr \wedge d\theta \end{aligned}$$

□

3.5 G_2 metrics

Let $f = f(r)$ and $g = g(r)$ be smooth functions to be determined, and consider the G_2-structure determined by the 3-form
$$\varphi = 2\sigma f g^2 d\theta + f^3\gamma.$$
Referring to the definitions of $d\theta$ and γ, this is an admissible 3-form in which $(f\alpha^i)$ forms part of an oON basis with respect to the metric defined by the inclusion $G_2 \subset SO(7)$. The function g represents a scaling for 1-forms on X, the function f a scaling on the fibres.

Remark 3.15. Note that a metric on $E \to X$ defined by the 3-form φ induces a metric on X, in the conformal class of the Urbantke metric.

Then
$$\begin{aligned} d\varphi &= 2\sigma(fg^2)'dr \wedge d\theta + f^3 d\gamma \\ &= 2\sigma(fg^2)'dr \wedge d\theta + f^3(\tfrac{1}{2}dr \wedge d\theta) \\ &= \left(2\sigma(fg^2)' + \tfrac{1}{2}f^3\right)dr \wedge d\theta \end{aligned}$$

since $d\theta$ is closed and $dr \wedge \gamma = 2y^j \alpha^j \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = 0$. So for φ to be closed, *i.e.* $d\varphi = 0$, we need

$$f^3 = -4\sigma(fg^2)'. \tag{5}$$

Notice that this is essentially the same equation as for closure in the Bryant-Salamon case.

Now in order to compute $*\varphi$, we put φ into canonical form. We first find a parametrisation of the curvature 2-forms in terms of an orthonormal basis of Λ^2_-. Let $Y = \sqrt{X_0}$ (this is well-defined since X_0 is positive definite and diagonalisable), so that $Y^{-2}X_0 = \mathrm{Id}$. Then $E^i = \sigma(Y^{-1})^i_j F^j$ is an orthonormal basis for the fibres of Λ^2_-:

$$E^i \wedge E^j = \sigma^2 (Y^{-1})^i_k (Y^{-1})^j_l F^k \wedge F^l = -2(Y^{-1})^i_k (Y^{-1})^j_l X_0^{kl} v_0 = -2\delta^{ij} v_0.$$

So in terms of this basis we have

$$\varphi = 2\sigma f g^2 d\theta + f^3\gamma = 2\sigma f g^2 \alpha^j \wedge F^j + f^3\gamma = 2fg^2 Y^i_j \alpha^j \wedge E^i + f^3\gamma.$$

Now if we write φ in the canonical form

$$\varphi = \theta^1 \wedge \theta^2 \wedge \theta^3 + \theta^1 \wedge \omega^1 + \theta^2 \wedge \omega^2 + \theta^3 \wedge \omega^3,$$

we have the following canonical form for its Hodge dual

$$*\varphi = -\frac{1}{6}\omega^i \wedge \omega^i - \frac{1}{2}\epsilon_{ijk}\theta^i \wedge \theta^j \wedge \omega^k.$$

Let us suppose now that $\theta^i = fY^i_j\alpha^j$ and $\omega^i = 2g^2 E^i$. Using the canonical form, we have

$$\begin{aligned}
\varphi &= f^3(Y^1_i \alpha^i \wedge Y^2_j \alpha^j \wedge Y^3_k \alpha^k) + 2fg^2 Y^i_j \alpha^j \wedge E^i \\
&= f^3 \epsilon_{ijk} Y^i_1 Y^j_2 Y^k_3 \alpha^1 \wedge \alpha^2 \wedge \alpha^3 + 2fg^2 Y^i_j \alpha^j \wedge E^i \\
&= f^3 (\det Y)\, \alpha^1 \wedge \alpha^2 \wedge \alpha^3 + 2fg^2 Y^i_j \alpha^j \wedge E^i \\
&= f^3\gamma + 2fg^2 Y^i_j \alpha^j \wedge E^i
\end{aligned}$$

since we can assume that Y has determinant 1. This agrees with the initial φ, so our canonical basis is correct, and we can use it to calculate

$$\begin{aligned}
*\varphi &= -\frac{1}{6}4g^4 E^i \wedge E^i - \frac{1}{2}2f^2g^2\epsilon_{ijk} Y^i_l \alpha^l \wedge Y^j_m \alpha^m \wedge E^k \\
&= -\frac{2}{3}g^4 E^i \wedge E^i - f^2g^2 \epsilon_{ijk} Y^i_l Y^j_m \alpha^l \wedge \alpha^m \wedge E^k.
\end{aligned}$$

We can translate this back in terms of the curvature forms to finally get

$$\begin{aligned}
*\varphi &= -\frac{2}{3}\sigma^2 g^4 (Y^{-2})^m_l F^l \wedge F^m - \sigma f^2 g^2 \epsilon_{ijk} Y^i_l Y^j_m \alpha^l \wedge \alpha^m \wedge Y^k_n (Y^{-2})^n_p F^p \\
&= -\frac{2}{3}g^4 (Y^{-2})^m_l F^l \wedge F^m - \sigma f^2 g^2 \epsilon_{lmn} (\det Y)\, \alpha^l \wedge \alpha^m \wedge (Y^{-2})^n_p F^p \\
&= -\frac{2}{3}g^4 (X_0^{-1}F)^m \wedge F^m - \sigma f^2 g^2 \epsilon_{ijk} (X_0^{-1}F)^k \wedge \alpha^j \wedge \alpha^i.
\end{aligned}$$

Remember that we could have chosen any volume form v, with associated matrix X, and the relationship was $X_0 = (\det X)^{-\frac{1}{3}} X$. So we have the following general form

$$*\varphi = -\frac{2}{3}g^4 (\det X)^{\frac{1}{3}} (X^{-1}F)^i \wedge F^i - \sigma f^2 g^2 \epsilon_{ijk} (\det X)^{\frac{1}{3}} (X^{-1}F)^i \wedge \alpha^j \wedge \alpha^k.$$

Now we will work out the co-closure conditions. Define the 2-forms $Z^i := (\det X)^{\frac{1}{3}} (X^{-1}F)^i$. Hit $*\varphi$ with d to get

$$\begin{aligned}
d*\varphi = &-\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i - \frac{2}{3}g^4 dZ^i \wedge F^i - \frac{2}{3}g^4 Z^i \wedge dF^i - \epsilon_{ijk}\, \sigma (f^2g^2)' dr \wedge Z^i \wedge \alpha^j \wedge \alpha^k \\
&- \sigma f^2 g^2 \epsilon_{ijk}\, dZ^i \wedge \alpha^j \wedge \alpha^k - \sigma f^2 g^2 \epsilon_{ijk}\, Z^i \wedge d\alpha^j \wedge \alpha^k + \sigma f^2 g^2 \epsilon_{ijk}\, Z^i \wedge \alpha^j \wedge d\alpha^k.
\end{aligned}$$

The third term vanishes since we are assuming $dF^i = 0$. The sixth and seventh terms are actually the same as each other because of symmetries. If we *suppose* that $dZ^i = 0$ then the second and fifth terms vanish. The fourth term is a 5-form proportional to the volume form of the fibre; this can't cancel with anything else and so we have to require that

$$(f^2g^2)' = 0. \tag{6}$$

This leaves us with

$$\begin{aligned}
d * \varphi &= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i - 2\sigma f^2 g^2 \epsilon_{ijk}\, Z^i \wedge (\epsilon_{jlm} y^l F^m) \wedge \alpha^k \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i - 2\sigma f^2 g^2 (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, y^l Z^i \wedge F^m \wedge \alpha^k \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i - 2\sigma f^2 g^2 (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, y^l (\det X)^{\frac{1}{3}} (X^{-1})^i_j F^j \wedge F^m \wedge \alpha^k \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i - 2\sigma f^2 g^2 (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, y^l (\det X)^{\frac{1}{3}} (X^{-1})^i_j (-2) X^{jm} v \wedge \alpha^k \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i + 4\sigma f^2 g^2 (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, y^l (\det X)^{\frac{1}{3}} \delta^{im} v \wedge \alpha^k \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i + 4\sigma f^2 g^2 (\delta_{kl}\delta_{ii} - \delta_{ki}\delta_{il})\, y^l \alpha^k \wedge (\det X)^{\frac{1}{3}} v \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i + 4\sigma f^2 g^2 (\delta_{kk} - \delta_{ki}\delta_{ik})\, y^k \alpha^k \wedge (\det X)^{\frac{1}{3}} v \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i + 2\sigma f^2 g^2 (1 - \delta_{ik})\, dr \wedge (\det X)^{\frac{1}{3}} (-\tfrac{1}{2}) (X^{ij})^{-1} F^j \wedge F^i \\
&= -\frac{2}{3}(g^4)' dr \wedge Z^i \wedge F^i - \frac{2}{3}\sigma f^2 g^2 dr \wedge Z^i \wedge F^i \\
&= -\frac{2}{3}\left[(g^4)' + \sigma f^2 g^2\right] dr \wedge Z^i \wedge F^i
\end{aligned}$$

where we used the fact that δ_{ik} vanishes for two-thirds of the possible permutations. Thus for φ to be co-closed, we must have

$$(g^4)' = -\sigma f^2 g^2. \tag{7}$$

Since equations (5), (6) and (7) are the same as those in the Bryant-Salamon construction (up to sign), we have a complete G_2 metric on the total space of E for the same solutions as before:

$$\sigma = -1 \qquad f = (1+r)^{-\frac{1}{4}} \qquad g = (1+r)^{\frac{1}{4}}$$

However, this was *only possible* because of our supposition that $dZ^i = 0$. Remember, we are working over a point for which '$d = \nabla$' (see discussion in Section 3.2). That is, we have $\nabla Z = \sum dZ^i \otimes P^i$, since we can assume that $\nabla P^i = 0$. Hence, requiring that $dZ^i = 0$ is the same as the condition $\nabla Z = 0$, *i.e.*

$$\nabla[(\det X)^{\frac{1}{3}} X^{-1} F] = 0. \tag{8}$$

This PDE is the field equation for the '4D gravity theory' defined by the Urbantke metric, as we shall discuss briefly later. We have now proved the theorem that was stated in [HKSS16]:

Theorem 3.16 (Herfray-Krasnov-Scarinci-Shtanov, [HKSS16])**.** *If ∇ is a definite connection satisfying $\nabla[(\det X)^{\frac{1}{3}} X^{-1} F] = 0$, there exist positive functions f, g such that the 3-form $\varphi = 2\sigma f g^2 d\theta + f^3 \gamma$ is positive, closed, and co-closed, hence defines a metric of G_2 holonomy. This metric is of Riemannian signature and is complete (in the fibre direction) for $\sigma = -1$.*

Remark 3.17. The sign $\sigma = \pm 1$ corresponds to the sign of the scalar curvature in the Bryant-Salamon case (and depends on our choice of orientation convention). It is necessary to give the metric Riemannian signature. Otherwise, the metric (though still complete in the fibre direction) will have signature $(3, 4)$. The metric induced on the base (which is in the conformal class that makes the curvature 2-forms F^i ASD) is always Riemannian though.

Remark 3.18. When $X^{ij} = \lambda\delta^{ij}$, there exists a metric with respect to which $F^i = \sigma\omega^i$, so the Bianchi identity tells us that $\nabla\omega^i = 0$, *i.e.* the connection is the ASD part of the Levi-Civita connection. Hence F^i is ASD, and so by Theorem 1.9, the metric on the base is Einstein. Moreover, $F^i = \sigma\omega^i$ also tells us that $W^- = 0$, and so the metric is also self-dual (*cf.* Fact 1). Then equation (8) is just the Bianchi identity, which always holds, and so the construction reduces to the Bryant-Salamon case.

Question 3.19. Is $(\det X)^{\frac{1}{3}}X^{-1}F$ just the curvature of some connection? *i.e.* Is the PDE just the Bianchi identity for a particular choice of connection?

3.6 Diffeomorphism-invariant $SO(3)$ gauge theories

The PDE in the above theorem is the Euler-Lagrange equation of a diffeomorphism-invariant theory of $SO(3)$-connections on X. Thus every solution to that theory can be lifted to a G_2-holonomy metric on the total space. We will briefly explain this relationship now.

Recall the equation defining the matrix X^{ij}

$$F^i \wedge F^j = -2X^{ij}v.$$

Remember that the volume form is defined only up to multiplication by a nowhere-vanishing function, so X^{ij} is only defined modulo this too. Let f be a function from symmetric 3×3 matrices to $\mathbb{R}$ satisfying

(i) gauge invariance: $f(OXO^T) = f(X)$ for $O \in SO(3)$,

(ii) homogeneity degree one: $f(\alpha X) = \alpha f(X)$ for $\alpha \in \mathbb{R}$.

Then $f(F^i \wedge F^j) := -2f(X^{ij})v$ is a well-defined gauge-invariant 4-form on X. So any such function leads to a diffeomorphism and gauge-invariant functional of $SO(3)$-connections

$$S_f[\nabla] = -\frac{1}{2}\int_X f(F^i \wedge F^j)$$

which is just the total volume of X computed using the volume form constructed from the curvature of the connection. The extrema of $S_f[\nabla]$ are connections satisfying the Euler-Lagrange equation

$$\nabla\left(\frac{\partial f}{\partial X^{ij}}F^j\right) = 0.$$

Remember that any choice of such an f defines a volume form, and hence also a metric in the conformal class of the Urbantke metric. The requirement that the connection satisfies its Euler-Lagrange equation constitutes a constraint on the metric. If we diagonalise X^{ij} (always possible since it's symmetric), we see that f must be a homogeneity degree one function of the eigenvalues.

In the Urbantke metric, we chose $f(X) = (\det X)^{\frac{1}{3}}$, which is perhaps the most mathematically natural choice, as it makes the triple of curvature forms equal to the canonical basis of ASD forms (1). Then $\frac{\partial f}{\partial X} = \frac{1}{3}(\det X)^{\frac{1}{3}}X^{-1}$, and so the Euler-Lagrange equation reduces to the PDE in Theorem 3.16. Hence every solution to the 'theory of gravity' defined by this choice of f can be lifted to a G_2 holonomy metric on the total space.

This theory of gravity is not general relativity, however, which requires the choice $f = (\operatorname{tr}\sqrt{X})^2$. This choice leads to Einstein metrics on the base, but the G_2 construction on the total space does not work in this case.

The relationship between solutions to the 4D and 7D theories is less surprising when you look at their action functionals. We saw in 1.24 that G_2 metrics can be considered critical points of a certain functional $S[\varphi]$, which is just the volume of the 7-manifold with respect to v_φ. For the 3-form φ associated to the Urbantke metric, this functional is a multiple of the functional $S[\nabla]$ for the $SO(3)$ gauge theory on the base. So it is no wonder that their solutions coincide!

Question 3.20. Can any (positive) definite connection be smoothly deformed through such connections to a critical point of $S[\nabla]$?

4 Calibrated submanifolds

The study of calibrated geometries began with Harvey and Lawson in their seminal paper of 1982 [HL82], and the area is now of great interest to both mathematicians and physicists, not least because calibrated submanifolds are believed to be an important component of theories such as mirror symmetry and M-theory.

We first review the relevant facts from calibrated geometry, and set up some notation. Calibrated submanifolds of a Riemannian manifold are volume-minimising in their homology class, and it is important to note that while being minimal is a second order condition, being calibrated is just first order.

Definition 4.1. A closed k-form α on M which satisfies $\alpha(e_1, \ldots, e_k) \leq 1$ for orthonormal tangent vectors $e_1, \ldots, e_k$ at any point $p \in M$ is called a *calibration*. An oriented k-dimensional subspace $V_p \subset T_pM$ is said to be *calibrated* if $\alpha(V_p) = 1$. A *calibrated submanifold* $L \subset M$ is a k-dimensional oriented submanifold whose tangent spaces are all calibrated. In fact this means that $\alpha|_L = \mathrm{vol}_L$, the volume form induced on L by the Riemannian metric on M.

Calibrated submanifolds occur naturally in manifolds with special holonomy. The holonomy group G of a Riemannian manifold (M, g) acts on the k-forms on each $T_p(M)$. Suppose α is a G-invariant k-form, then rescale so that $\alpha|_P \leq \mathrm{vol}_P$ for all oriented k-planes P, with equality satisfied for at least one P. Then that P is calibrated, and so are $g \cdot P$ for any $g \in G$ (since α is G-invariant). We get a parallel k-form on M which is just α at each point, and since $\nabla\alpha = 0$ it must be closed. So α is a calibration on M, which calibrates a lot of its tangent planes, hence there is a high probability of M having calibrated submanifolds.

There are four main examples of calibrated geometries:

(i) Complex submanifolds L^{2k} of a Kählermanifold M. The calibration is $\alpha = \frac{\omega^k}{k!}$, where ω is the Kählerform on M. Kählermanifolds have holonomy contained in $U(n)$, and the submanifolds come in all even real dimensions.

(ii) Special Lagrangian submanifolds L^n (with phase $e^{i\theta}$) of a Calabi-Yau manifold M. The calibration is $\mathrm{Re}(e^{i\theta}\Omega)$, where Ω is the holomorphic $(n, 0)$-form on M. Calabi-Yau manifolds have holonomy contained in $SU(n)$, and special Lagrangians are always half-dimensional (but there is an S^1 family of calibrations for each M, because of the $e^{i\theta}$ freedom in a choice of Ω). NB Calabi-Yau manifolds are Kähler, so also contain the calibrated complex submanifolds.

(iii) Associative and coassociative submanifolds L^3, L^4 of a G_2-manifold M^7. The calibrations are φ and $*\varphi$ respectively, where φ is the fundamental 3-form associated to the G_2-structure. G_2-manifolds have holonomy contained in G_2, and the submanifolds only come in dimensions 3 and 4.

(iv) Cayley submanifolds L^4 of a Spin(7) manifold M^8. The calibration is Φ, which is the fundamental 4-form associated to the Spin(7)-structure. Spin(7) manifolds obviously have holonomy contained in Spin(7), and the submanifolds only come in dimension 4.

The condition that $\alpha|_L = \mathrm{vol}_L$ is often tricky to check, so we now describe some equivalent characterisations:

(i) Complex submanifolds are characterised by their tangent spaces being invariant under the action of the complex structure J.

(ii) Harvey and Lawson showed in [HL82] that (up to orientation), L is special Lagrangian of phase $\mathrm{e}^{i\theta}$ iff $\omega|_L = 0$ and $\mathrm{Im}(e^{i\theta}\Omega)|_L = 0$. The first condition gives Lagrangians (involving only the symplectic structure); adding the second makes it *special* (using the metric too).

(iii) Harvey and Lawson also showed in [HL82] that L^4 is coassociative iff $\varphi|_{L^4} = 0$, and L^3 is associative iff its tangent spaces are preserved by the cross product (which comes from identifying T_pM with the imaginary octonions). It is shown in [KM] that the associative condition is equivalent to $u \lrcorner v \lrcorner w \lrcorner * \varphi = 0$ for all tangent vectors $u, v, w \in T_pL^3$.

(iv) It is again shown in [HL82] that L^4 is Cayley iff its tangent space is preserved by a three-fold vector cross product.

We will briefly recall the bundle construction of Harvey and Lawson for special Lagrangians, since later results for G_2-manifolds are closely related to this. Special Lagrangians naturally live in Calabi-Yau manifolds, which are in particular symplectic. A famously symplectic manifold is the cotangent bundle, and since $T^*\mathbb{R}^n = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$, we know that $T^*\mathbb{R}^n$ is also Calabi-Yau.

Fact 2. *The conormal bundle N^*M^p of a submanifold $M^p \subset \mathbb{R}^n$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$.*

Motivated by this classical result, Harvey and Lawson found the conditions on the immersion $M^p \subset \mathbb{R}^n$ that make N^*M^p a *special* Lagrangian submanifold of $T^*\mathbb{R}^n$.

Theorem 4.2 (Harvey-Lawson, [HL82]). *The conormal bundle N^*M^p is special Lagrangian in $T^*\mathbb{R}^n$ (with phase i^q, $q = n - p$) iff M^p is austere in $\mathbb{R}^n$.*

Definition 4.3. M is *austere* if all the odd degree symmetric polynomials in the eigenvalues of A^ν vanish for all normal vector fields ν on M, where A^ν is the second fundamental form for the immersion of M in $\mathbb{R}^n$. Equivalently, all eigenvalues of A^ν occur in pairs of opposite signs.

Remark 4.4. The first symmetric polynomial is the trace, and the vanishing of the trace of the second fundamental form is exactly the minimality condition. In fact, for $p = 1, 2$ this is the only condition, but for $p \geq 3$ austere is much stronger than minimal.

Relatively little is known still about calibrated submanifolds in general Calabi-Yau, G_2, and Spin(7) manifolds (even non-compact ones), but the cases of $\mathbb{C}^n$, $\mathbb{R}^7$, and $\mathbb{R}^8$ are better understood, and provide useful local models.

4.1 Associative and coassociative submanifolds

Ionel, Karigiannis and Min-Oo [IKMO05] generalised the Harvey-Lawson bundle construction of special Lagrangians in $\mathbb{C}^n$ to construct associative and coassociative submanifolds of $\mathbb{R}^7$, by considering it as the total space of a vector bundle and then taking appropriate subbundles defined over particular surfaces in $\mathbb{R}^4$.

As we already know, $\mathbb{R}^7$ regarded as the space of anti-self-dual 2-forms on $\mathbb{R}^4$ has a naturally defined G_2-structure. Candidates for associative and coassociative submanifolds can be obtained by considering the subbundle defined by restricting this bundle to an immersed surface $\Sigma^2 \subset \mathbb{R}^4$.

Since calibrated submanifolds are always minimal, and the bundle directions have trivial second fundamental form, the surfaces Σ^2 must be at least minimal in $\mathbb{R}^4$ (just like the austere submanifolds of Harvey and Lawson). In fact, we have the following.

Theorem 4.5 (Ionel-Karigiannis-Min-Oo, [IKMO05]). *The naturally defined rank 2 subbundle of $\Lambda^2_-(\mathbb{R}^4)|_{\Sigma^2}$ is coassociative iff the immersion of Σ^2 in $\mathbb{R}^4$ is a solution of exactly one half of the real isotropic minimal surface equation (also called superminimal).*

This construction produces examples of coassociatives which are actually just complex submanifolds of a copy of $\mathbb{C}^3 = \mathbb{R}^6$ living in $\mathbb{R}^7$. It's not completely boring, because which copy of $\mathbb{C}^3$ they are sitting in depends on the immersion of Σ in $\mathbb{R}^4$. They did manage to find more interesting coassociatives though, in G_2-manifolds that are vector bundles over a compact base, as we shall see in the next section. They also obtained the following, perhaps surprising, result.

Theorem 4.6 (Ionel-Karigiannis-Min-Oo, [IKMO05]). *The naturally defined rank 1 subbundle of* $\Lambda^2_-(\mathbb{R}^4)|_{\Sigma^2}$ *is associative iff* Σ^2 *is minimal in* $\mathbb{R}^4$.

Remark 4.7. The interesting phenomenon that special Lagrangian and coassociative submanifolds are harder to construct in this way (requiring a more-than-minimal base) than associatives (and in fact also Cayley submanifolds in $\mathbb{R}^8$) seems to be reflected in their deformation theory. Special Lagrangians and coassociatives are defined by the vanishing of differential forms, and have a nice, unobstructed local deformation theory [McL98]. Whereas associative (and Cayley) submanifolds are defined by their tangent spaces being preserved by a cross product operation, and have a complicated and obstructed deformation theory.

4.2 Calibrated submanifolds of the Bryant-Salamon metrics

Karigiannis and Min-Oo [KMO05] further generalised the Harvey-Lawson construction to find calibrated submanifolds in non-flat, non-compact manifolds with complete special holonomy metrics that are bundles over a *compact base*. In the case of T^*S^n with the Stenzel Calabi-Yau metric, they found that the conormal bundle of an immersed submanifold $X \subset S^n$ is again special Lagrangian (with respect to some phase which depends on the codimension of X in S^n) iff the submanifold is austere. This is somewhat surprising, since the complex structure on T^*S^n is obtained very differently from that on $\mathbb{C}^n = T^*\mathbb{R}^n$ (by identifying it with a complex quadric hypersurface in $\mathbb{C}^{n+1}$).

We will now look in more detail at the construction of calibrated submanifolds of $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ with the Bryant-Salamon complete G_2 metrics (though we note that the constructions which follow also work in the incomplete cases). As in the previous examples, these are constructed as subbundles over immersed surfaces in the base. We show that the surface is required to be minimal in the associative case, and (properly oriented) real isotropic in the coassociative case.

Remark 4.8. Even though the surface Σ is only immersed in the base, the resulting calibrated submanifold is actually embedded!

4.2.1 The second fundamental form

We set up some useful notation for our local calculations on immersed surfaces $\Sigma \subset X$. Let e_1, e_2 and ν_1, ν_2 be local orthonormal frames of tangent and normal vector fields to Σ near a point $x_0 \in X$, respectively. Let ∇ be the Levi-Civita connection on X, and $()^T$ and $()^N$ denote orthogonal projections onto $T(\Sigma)$ and $N(\Sigma)$ respectively. As in the [HKSS16] construction, by parallel transporting via the induced tangent and normal connections, we can assume that our local frames have been chosen so that at a point $x_0 \in X$

$$(\nabla_{e_i} e_j)|^T_{x_0} = 0 \qquad \text{and} \qquad (\nabla_{e_i} \nu_j)|^N_{x_0} = 0. \tag{9}$$

Now recall that for ν a normal vector field, the *second fundamental form* is the linear operator

$$A^\nu : T(\Sigma) \to T(\Sigma)$$
$$A^\nu : \xi \mapsto A^\nu(\xi) = (\nabla_\xi \nu)^T$$

for $\xi \in T(\Sigma)$. We use the Harvey-Lawson sign convention for A^ν, though all of our results are independent of this choice anyway.

Proposition 4.9. A^ν *is a symmetric operator (hence diagonalisable).*

Proof. $\langle e_i, A^\nu(e_j)\rangle = \langle e_i, (\nabla_{e_j}\nu)^T\rangle = \langle e_i, \nabla_{e_j}\nu\rangle = -\langle \nabla_{e_j} e_i, \nu\rangle$

$$\begin{aligned} &= -\langle \nabla_{e_i} e_j, \nu\rangle + \langle [e_i, e_j], \nu\rangle \\ &= -\langle \nabla_{e_i} e_j, \nu\rangle \\ &= \langle e_j, \nabla_{e_i}\nu\rangle \\ &= \langle e_j, (\nabla_{e_i}\nu)^T\rangle \\ &= \langle e_j, A^\nu(e_i)\rangle \end{aligned}$$

since the Lie bracket of two tangent vector fields is another tangent vector field, so $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i$ is orthogonal to ν. ☐

For simplicity, we will henceforth use the notation $A^k_{ij} := A^{\nu_k}_{ij} = \langle A^{\nu_k}(e_i), e_j\rangle = A^k_{ji}$.

Fact 3. *Recall that the trace of the second fundamental form with respect to the metric is the mean curvature. Recall also that the vanishing of the mean curvature is an equivalent condition for minimality of a submanifold. These facts will play a part later.*

4.2.2 Submanifolds

As before, let (X, g) be an oriented self-dual Einstein 4-manifold, and $M = \Lambda^2_- T^*X$ be the bundle of anti-self-dual 2-forms on X. Suppose Σ is an oriented surface immersed in X, with the induced metric. Let e_1, e_2 and ν_1, ν_2 be local orthonormal frames of tangent and normal vector fields to Σ respectively. Then the dual covector fields (e^1, e^2, ν^1, ν^2) form an adapted oON moving coframe of X along the surface Σ.

Locally, the restriction of anti-self-dual 2-forms to Σ can be written

$$\Lambda^2_- T^*X|_\Sigma = \mathrm{span}(\omega^1, \omega^2, \omega^3)$$

where $\omega^1 = e^1 \wedge e^2 - \nu^1 \wedge \nu^2$, $\omega^2 = e^1 \wedge \nu^1 - \nu^2 \wedge e^2$ and $\omega^3 = e^1 \wedge \nu^2 - e^2 \wedge \nu^1$.

Now ω^1 is globally well-defined, and independent of the local frame, so we can define the line bundle $L \subset \Lambda^2_- M$ over Σ to be the span of ω^1. Similarly, $L^\perp = \{\omega \in \Lambda^2_- | \omega \perp \omega^1\}$ is the rank 2 bundle over Σ orthogonal to L (with respect to the Bryant-Salamon metric on $\Lambda^2_- X$), and is locally spanned by ω^2 and ω^3. We then have the splitting

$$\Lambda^2_- T^*X|_\Sigma = L \oplus L^\perp.$$

The total spaces of L and $L^\perp$ are 3 and 4-dimensional submanifolds of $\Lambda^2_- X$, and so are candidates for associative and coassociative submanifolds. [KMO05] determined conditions on the second fundamental form of Σ for this to be the case. Before stating and proving their result, we prove a couple of propositions which will simplify later calculations.

Proposition 4.10. *We have the following expressions for the covariant derivatives of the dual coframe fields e^i, ν^j:*

$$\nabla_{e_i} e^j = -\sum_k A^k_{ij} \nu^k \qquad\qquad \nabla_{e_i} \nu^j = \sum_k A^j_{ik} e^k$$

Proof. The coframe fields satisfy

$$e^i(e_j) = \delta^i_j \qquad \nu^i(\nu_j) = \delta^i_j \qquad e^i(\nu_j) = 0 \qquad \nu^i(e_j) = 0$$

so we get $(\nabla_{e_i} e^j)(\nu_k) + e^j(\nabla_{e_i}\nu_k) = \nabla_{e_i}(e^j(\nu_k)) = 0$, which combined with the identities (9) gives us

$$(\nabla_{e_i} e^j)(\nu_k) = -e^j(\nabla_{e_i}\nu_k) = -e^j\left(\sum_l \langle \nabla_{e_i}\nu_k, e_l\rangle e_l\right) = -e^j \sum_l A^k_{il} e_l = -A^k_{ij}$$

and

$$(\nabla_{e_i}\nu^j)(e_k) = -\nu^j(\nabla_{e_i} e_k) = -\nu^j\left(\sum_l \langle \nabla_{e_i} e_k, \nu_l\rangle \nu_l\right) = \nu^j\left(\sum_l \langle e_k, \nabla_{e_i}\nu_l\rangle \nu_l\right) = \nu^j \sum_l A^l_{ik}\nu_l = A^j_{ik}$$

☐

Proposition 4.11. *We have the following expressions for the covariant derivatives of $\omega^1, \omega^2, \omega^3$ in the e_1, e_2 directions at a point $x_0 \in X$.*

$$\begin{aligned}
\nabla_{e_i}\omega^1 &= (A^2_{i1} - A^1_{i2})\omega^2 + (-A^1_{i1} - A^2_{i2})\omega^3 \\
\nabla_{e_i}\omega^2 &= (A^1_{i2} - A^2_{i1})\omega^1 \\
\nabla_{e_i}\omega^3 &= (A^2_{i2} + A^1_{i1})\omega^1
\end{aligned}$$

Proof. I will expand the second identity (the rest are similar)

$$\begin{aligned}
\nabla_{e_i}\omega^2 &= (\nabla_{e_i}e^1) \wedge \nu^1 + e^1 \wedge (\nabla_{e_i}\nu^1) - (\nabla_{e_i}\nu^2) \wedge e^2 - \nu^2 \wedge (\nabla_{e_i}e^2) \\
&= (-A^1_{i1}\nu^1 - A^2_{i1}\nu^2) \wedge \nu^1 + e^1 \wedge (A^1_{i1}e^1 + A^1_{i2}e^2) \\
&\quad - (A^2_{i1}e^1 + A^2_{i2}e^2) \wedge e^2 - \nu^2 \wedge (-A^1_{i2}\nu^1 - A^2_{i2}\nu^2) \\
&= (A^1_{i2} - A^2_{i1})(e^1 \wedge e^2 - \nu^1 \wedge \nu^2)
\end{aligned}$$

□

Theorem 4.12 (Karigiannis-Min-Oo, [KMO05]). *L is an associative submanifold of $\Lambda^2_- T^*M$ equipped with the Bryant-Salamon G_2 metric iff Σ is a minimal surface in X. The bundle $L^\perp$ is coassociative iff Σ is a (properly oriented) real isotropic ("superminimal") surface in M.*

Proof. First the associative case. For L to be an associative submanifold, we must have T_pL associative in $T_p(\Lambda^2_- X) \cong \mathbb{R}^7$ for every $p \in L$. We will find the conditions on the immersion $\Sigma \subset X$ for this to be the case. At a point $t_1\omega^1 \in L$, the following form a basis of $T_{t_1\omega^1}L$

$$\begin{aligned}
E_i &= \beta_i + t_1(\nabla_{e_i}\omega^1)_{\mathcal{V}}, && i = 1, 2 \\
F_1 &= \alpha_1
\end{aligned}$$

where $(\ )_{\mathcal{V}}$ represents the vertical projection. Remember that $\beta_i = (e_i, 0)$, $\alpha_i = (0, \omega^i)$, and $\omega^1 = e^1 \wedge \nu^1 - \nu^2 \wedge e^2$. By Proposition 4.11, we have that locally

$$(\nabla_{e_i}\omega^1)_{\mathcal{V}} = (A^2_{i1} - A^1_{i2})\alpha_2 + (-A^1_{i1} - A^2_{i2})\alpha_3.$$

Now, we know that L is associative iff $E_1 \lrcorner E_2 \lrcorner F_1 \lrcorner * \varphi$ vanishes at all points of L. So here goes... (β^{ij} denotes $\beta^i \wedge \beta^j$)

$$\begin{aligned}
F_1 \lrcorner * \varphi &= \alpha_1 \lrcorner \Big[g^4(\beta^{1234}) - f^2g^2\alpha^2 \wedge \alpha^3 \wedge (\beta^{12} - \beta^{34}) \\
&\quad - f^2g^2\alpha^3 \wedge \alpha^1 \wedge (\beta^{13} - \beta^{42}) - f^2g^2\alpha^1 \wedge \alpha^2 \wedge (\beta^{14} - \beta^{23}) \Big] \\
&= -f^2g^2 \left(\alpha^2 \wedge (\beta^{14} - \beta^{23}) - \alpha^3 \wedge (\beta^{13} - \beta^{42}) \right)
\end{aligned}$$

which then eats $E_2 = \beta_2 + t_1\left((A^2_{21} - A^1_{22})\alpha_2 + (-A^1_{21} - A^2_{22})\alpha_3\right)$ to become

$$\begin{aligned}
E_2 \lrcorner F_1 \lrcorner * \varphi = -f^2g^2 \left(\alpha^2 \wedge \beta^3 + \alpha^3 \wedge \beta^4 \right) - t_1 f^2g^2 \big[&(A^2_{21} - A^1_{22})(\beta^1 \wedge \beta^4 - \beta^2 \wedge \beta^3) \\
&+ (A^1_{21} + A^2_{22})(\beta^1 \wedge \beta^3 - \beta^4 \wedge \beta^2) \big]
\end{aligned}$$

which in turn eats $E_1 = \beta_1 + t_1\left((A^2_{11} - A^1_{12})\alpha_2 + (-A^1_{11} - A^2_{12})\alpha_3\right)$ and becomes

$$\begin{aligned}
E_1 \lrcorner E_2 \lrcorner F_1 \lrcorner * \varphi &= -f^2g^2t_1 \big[(A^2_{21} - A^1_{22})\beta^4 + (A^1_{21} + A^2_{22})\beta^3 + (A^211 - A^1_{12})\beta^3 - (A^1_{11} + A^2_{12})\beta^4 \big] \\
&= -f^2g^2t_1 \big[(-A^1_{22} - A^1_{11})\beta^4 + (A^2_{22} + A^2_{11})\beta^3 \big]
\end{aligned}$$

since the second fundamental form is symmetric. Now f and g are positive functions, so for $*\varphi = 0$ on all of L (*i.e.* for all t_1) we must have

$$-A^1_{22} - A^1_{11} = 0 \qquad \text{and} \qquad A^2_{22} + A^2_{11} = 0$$

i.e. $\mathrm{tr}(A^{\nu_1}) = \mathrm{tr}(A^{\nu_2}) = 0$, *i.e.* Σ a minimal surface in X.

Now for the coassociatives. $L^\perp$ has a basis of tangent vectors at a point $t_2\omega^2 + t_3\omega^3$ given by

$$\begin{aligned} E_i &= \beta_i + t_2(\nabla_{e_i}\omega^3)_{\mathcal{V}} + t_3(\nabla_{e_i}\omega^3)_{\mathcal{V}}, && i = 1,2 \\ F_j &= \alpha_j. && j = 2,3 \end{aligned}$$

Again by Proposition 4.11, we have the vertical correction terms

$$\begin{aligned} (\nabla_{e_i}\omega^2)_{\mathcal{V}} &= (A^1_{i2} - A^2_{i1})\alpha_1 \\ (\nabla_{e_i}\omega^3)_{\mathcal{V}} &= (A^2_{i2} + A^1_{i1})\alpha_1 \end{aligned}$$

Remember that $L^\perp$ is coassociative iff $\varphi|_{L^\perp} = 0$, on each tangent space, so this is what we will check. Define the orthogonal normal vectors $\nu = t_2\nu_1 + t_3\nu_2$ and $\nu^\perp = -t_3\nu_1 + t_2\nu_2$, so that

$$\begin{aligned} E_1 &= \beta_1 + (A^\nu_{12} - A^{\nu^\perp}_{11})\alpha_1 \\ E_2 &= \beta_2 + (A^\nu_{22} - A^{\nu^\perp}_{12})\alpha_1 \end{aligned}$$

We first calculate $\varphi(E_1, E_2, \cdot)$:

$$\begin{aligned} E_1 \lrcorner \varphi = E_1 \lrcorner \big(& f^3\alpha^1 \wedge \alpha^2 \wedge \alpha^3 + g^2 f\alpha^1 \wedge (\beta^{12} - \beta^{34}) \\ & + g^2 f\alpha^2 \wedge (\beta^{13} - \beta^{42}) + g^2 f\alpha^3 \wedge (\beta^{14} - \beta^{23})\big) \\ = {} & g^2 f \left(\alpha^1 \wedge \beta^2 + \alpha^2 \wedge \beta^3 + \alpha^3 \wedge \beta^4\right) + (A^\nu_{12} - A^{\nu^\perp}_{11}) \left(f^3\alpha^2 \wedge \alpha^3 + g^2 f(\beta^{12} - \beta^{34})\right). \end{aligned}$$

Feed this E_2, to get

$$E_2 \lrcorner E_1 \lrcorner \varphi = g^2 f\alpha^1 + (A^\nu_{12} - A^{\nu^\perp}_{11}) g^2 f\beta^1 + (A^\nu_{22} - A^{\nu^\perp}_{12}) g^2 f\beta^2.$$

Now since $F_j = \alpha_j$, we always have $\varphi(E_1, E_2, F_j) = 0$ for $j = 2,3$. So we just need to check $\varphi(F_2, F_3, E_i)$ for $i = 1,2$.

$$\begin{aligned} F_2 \lrcorner \varphi &= f^3\alpha^1 \wedge \alpha^3 + g^2 f(\beta^{13} - \beta^{42}) \\ F_3 \lrcorner F_2 \lrcorner \varphi &= f^3\alpha^1 \\ E_1 \lrcorner F_3 \lrcorner F_2 \lrcorner \varphi &= f^3(A^\nu_{12} - A^{\nu^\perp}_{11}) \\ E_2 \lrcorner F_3 \lrcorner F_2 \lrcorner \varphi &= f^3(A^\nu_{22} - A^{\nu^\perp}_{12}) \end{aligned}$$

Since v is always positive, for the tangent space at (x_0, t_2, t_3) to be coassociative, we need

$$A^\nu_{12} - A^{\nu^\perp}_{11} = 0 \qquad \text{and} \qquad A^\nu_{22} - A^{\nu^\perp}_{12} = 0.$$

We get another two equations by asking that the tangent space at $(x_0, -t_3, t_2)$ is coassociative. This is effectively sending t_2 to $-t_3$ and t_3 to t_2, *i.e.* $\nu \mapsto \nu^\perp$ and $\nu^\perp \mapsto -\nu$. So we get

$$A^{\nu^\perp}_{12} + A^\nu_{11} = 0 \qquad \text{and} \qquad A^{\nu^\perp}_{22} + A^\nu_{12} = 0.$$

The surfaces defined by these four conditions are called *isotropic* (with negative orientation), or *superminimal*. We will talk about these in the next section. □

4.2.3 Superminimal surfaces

It is a truth universally acknowledged, that an oriented rank 2 real vector bundle in possession of a Riemannian metric on each fibre, must come equipped with a natural almost complex structure (ACS), J. For v_1, v_2 an oON basis in a given fibre, define it via $Jv_1 = v_2$ and $Jv_2 = -v_1$. Changing the orientation changes the sign of J.

The rank 2 vector bundles $T(\Sigma)$ and $N(\Sigma)$ inherit Riemannian metrics from the immersion of Σ in X, and a choice of orientation on one induces one on the other (of course if Σ is already oriented there is no choice).

Definition 4.13. A minimal surface Σ in X is called *real isotropic* if $A^{J\nu} = \pm J A^{\nu}$ for any normal vector field ν.

The conditions for coassociativity that we found showed that the matrix A^{ν} determines the matrix $A^{\nu^\perp}$, and the four equations can be written in the following matrix equation

$$\begin{pmatrix} A^{\nu^\perp}_{11} & A^{\nu^\perp}_{12} \\ A^{\nu^\perp}_{21} & A^{\nu^\perp}_{22} \end{pmatrix} = \begin{pmatrix} A^{\nu}_{12} & A^{\nu}_{22} \\ -A^{\nu}_{11} & -A^{\nu}_{12} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^{\nu}_{11} & A^{\nu}_{12} \\ A^{\nu}_{21} & A^{\nu}_{22} \end{pmatrix}. \tag{10}$$

The natural ACS we described above can be written $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $A^{\nu^\perp} = A^{J\nu}$, the conditions (10) for coassociativity reduce to the equation

$$A^{J\nu} = -JA^{\nu}.$$

So only half of the real isotropic surfaces in X give rise to coassociative submanifolds of $\Lambda^2_- T^*X$ (in fact the other half are submanifolds of $\Lambda^2_+ T^*X$).

Remark 4.14. In the above condition, the first J is the ACS on $N(\Sigma)$, whereas the second J is the ACS on $T(\Sigma)$. Notice that if we change the orientation of $T(\Sigma)$, the orientation on $N(\Sigma)$ changes, and so do the signs of both J's, leaving the equation invariant. So the condition is independent of the choice of orientation.

The equations encoded in (10) imply that $A^{\nu}_{11} + A^{\nu}_{22} = A^{\nu^\perp}_{11} + A^{\nu^\perp}_{22} = 0$. Since ν and $\nu^\perp$ are a basis for $N(\Sigma)$ at every point, this means that $\mathrm{tr}(A) = 0$, *i.e.* Σ is minimal in X. However, just as the austere condition for constructing special Lagrangians is stronger than minimality, so too is the condition $A^{J\nu} = -JA^{\nu}$, hence the name: *superminimal* surfaces.

4.3 Further thoughts

Conjecture 4.15. *The conditions for calibrated subbundles over immersed surfaces are the same in the [HKSS16] SO(3)-bundle construction as in the Bryant-Salamon case.*

I think that the above conjecture is highly likely to be true, and nearly managed to prove it, but ran out of time to sort out the messy calculations. The subbundles have to be defined as the spans of the *orthonormal* basis $E^i = \sigma(Y^{-1})^i_j F^j$ for the fibres of Λ^2_-.

Mclean proved the following remarkable result for compact calibrated submanifolds.

Theorem 4.16 (Mclean, [McL98]). *Suppose X is a compact calibrated submanifold of M, a manifold with special holonomy. A neighbourhood of X in M is naturally isomorphic to a neighbourhood of the zero-section of the normal bundle $N(X)$ of X in M. In the case of the special Lagrangian calibration $N(X)$ is isomorphic to $T^*(X)$, and for coassociatives $N(X) \cong \Lambda^2_-(X)$.*

In our earlier examples, the zero-section and the fibres are calibrated, so this theorem suggests that these constructions could potentially be used as a local model for the intersection of compact calibrated submanifolds in manifolds with special holonomy. This prompts the following question:

Question 4.17. Must the intersection of two compact coassociative submanifolds always be a superminimal immersion in those submanifolds?

Another interesting question prompted by McClean's theorem (motivated by a well-known result from symplectic geometry) is the following:

Question 4.18. Weinstein showed [Wei71] that a neighbourhood of a Lagrangian L in a symplectic manifold is symplectomorphic to a neighbourhood of the zero-section in T^*L. McClean says that a neighbourhood of a coassociative X is *isomorphic* to a neighbourhood of the zero-section in $\Lambda^2_- X$, but is it diffeomorphic?

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